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SOME EMBEDDINGS INTO THE TOTAL MORREY SPACES ASSOCIATED WITH THE DUNKL OPERATOR ON THE REAL LINE

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Abstract. On the real line, the Dunkl operators are differential-difference operators associated with the reflection group Z_2 on R. We consider the generalized shift operator, associated with the Dunkl operator $\Lambda_\alpha(f)(x)=\frac{d}{dx}f(x)+\frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right)$. We study some embeddings into the total Morrey space (D-total Morrey space) $L_{p,\lambda,\mu}(R)$, $0\le\lambda,\mu<2\alpha+2$ associated with the Dunkl operator on R. These spaces generalize the Morrey space associated with the Dunkl operator on R (D-Morrey space) so that $L_{p,\lambda}(R)\equiv L_{p,\lambda,\lambda}(R)$ and the modified Morrey spaces associated with the Dunkl operator on R so that $\tilde{L}_{p,\lambda}(R)\equiv L_{p,\lambda,0}(R)$.

Keywords: Dunkl operator, generalized translation operator, total-Morrey space.

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1. Introduction

On the real line, the Dunkl operators are differential-difference operators introduced in 1989 by Dunkl [1] and are denoted by Λ_α, where α is a real parameter $>-1/2$. These operators are associated with the reflection group Z_2 on R. The Dunkl kernel E_α is used to define the Dunkl transform $\Im_\alpha$ which was introduced by Dunkl in [2]. Rosler in [13] shows that the Dunkl kernels verify a product formula. This allows us to define the Dunkl translation $\tau_x, x\in R$. As a result, we have the Dunkl convolution.

Morrey spaces, introduced by C.B. Morrey [7], play important roles in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In [3] Guliyev introduce a variant of Morrey spaces called total Morrey spaces $L_{p,\lambda,\mu}(R)$, $0<p<\infty$, $\lambda\in R$ and $\mu\in R$, see also, [10].

In the present work, we give basic properties of the total Morrey space (Dtotal Morrey space) $L_{p,\lambda,\mu}(R)$, $0 \le \lambda, \mu < 2\alpha + 2$ associated with the Dunkl operator on R and study some embeddings into the total Morrey space $L_{p,\lambda,\mu}(R)$. These spaces generalize the Morrey space associated with the Dunkl operator on R (D-Morrey space) so that $L_{p,\lambda}(R) \equiv L_{p,\lambda,\lambda}(R)$ and the modified Morrey spaces associated with the Dunkl operator on R so that $\tilde{L}_{p,\lambda}(R) \equiv L_{p,\lambda,0}(R)$.

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In section 3, we give some embeddings into the total D-Morrey spaces.

Finally, we make some conventions on notation. By A $\lesssim$ B we mean that A $\le$ CB with some positive constant C independent of appropriate quantities. If A $\lesssim$ B and B $\lesssim$ A, we write A $\approx$ B and say that A and B are equivalent.

2. Preliminaries

For a real parameter $\alpha \ge -1/2$, we consider the Dunkl operator, associated with the reflection group Z_2 on R:

$$\Lambda_\alpha(f)(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x) - f(-x)}{2}\right).$$

Note that $\Lambda_{-1/2} = d/dx$.

For $\alpha \ge -1/2$ and $\lambda \in C$, the initial value problem:

$$\Lambda_\alpha(f)(x) = \lambda f(x), \quad f(0) = 1, \quad x \in R$$

has a unique solution $E_\alpha(\lambda x)$ called Dunkl kernel [1, 11, 14] and given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)} j_{\alpha+1}(i\lambda x), \quad x \in R,$$

where j_α is the normalized Bessel function of the first kind and order α [15], defined by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n+\alpha+1)}, \quad z \in C.$$

We can write for $x \in R$ and $\lambda \in C$ (see Rösler [13], p. 295)

$$E_\alpha(-i\lambda x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_{-1}^{1}(1-t^2)^{\alpha-1/2}(1-t)e^{i\lambda x t}\, dt.$$

Note that $E_{-1/2}(\lambda x) = e^{\lambda x}$.

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on R, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1}|x|^{2\alpha+1}dx.$$

For every $1 \le p \le \infty$, we denote by $L_p = L_p(d\mu_\alpha)$ the spaces of complex-valued functions f, measurable on R such that

$$\|f\|_{p,\alpha} \equiv \|f\|_{L_{p,\alpha}} = \left(\int_R |f(x)|^p d\mu_\alpha(x)\right)^{1/p} < \infty \quad if \quad p \in [1,\infty),$$

and

$$\|f\|_{\infty,\alpha} = ess\sup_{x\in R}|f(x)| \quad if \quad p = \infty.$$

For $1 \le p < \infty$ we denote by $WL_{p,\alpha}$, the weak $L_{p,\alpha}$ spaces defined as the set of locally integrable functions $f(x),\ x \in R$ with the finite norm

$$\|f\|_{WL_{p,\alpha}} = \sup_{r>0} r\left(\mu_\alpha\{x \in R : |f(x)| > r\}\right)^{1/p}.$$

Note that

$$L_{p,\alpha} \subset WL_{p,\alpha} \text{ and } \|f\|_{WL_{p,\alpha}} \le \|f\|_{L_{p,\alpha}} \text{ for all } f \in L_{p,\alpha}.$$

The Dunkl kernel gives rise to an integral transform, called Dunkl transform on R, which was introduced and studied in [5].

The Dunkl transform $\mathfrak{I}_\alpha$ of a function $f \in L_{1,\alpha}(R)$, is given by

$$\mathfrak{I}_\alpha f(\lambda) := \int_R E_\alpha(-i\lambda x) f(x) d\mu_\alpha(x), \quad \lambda \in R.$$

Here the integral makes sense since $|E_\alpha(ix)| \le 1$ for every $x \in R$ [13], p. 295.

Note that $\mathfrak{I}_{-1/2}$ agrees with the Fourier transform $\mathfrak{I}$, given by:

$$\mathfrak{I}f(\lambda) := (2\pi)^{-1/2}\int_R e^{-i\lambda x} f(x)dx, \quad \lambda \in R$$

Notation. For all $x, y, z \in R$, we put

$$W_\alpha(x,y,z) = \left(1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}\right)\Delta_\alpha(x,y,z)$$

where

$$\sigma_{x,y,z} = \begin{cases} \dfrac{x^2+y^2-z^2}{2xy} & if \quad x,y \in R\setminus 0, \\ 0 & otherwice \end{cases}$$

and Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x,y,z)=\begin{cases} d_\alpha \dfrac{\left(\left[(|x|+|y|)^2-z^2\right]\left[z^2-(|x|-|y|)^2\right]\right)^{\alpha-1/2}}{|xyz|^{2\alpha}} & if \quad |z|\in A_{x,y}, \\ 0 & otherwice, \end{cases}$$

where $d_\alpha = (\Gamma(\alpha+1))^2 / \left(2^{\alpha-1}\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)\right)$ and $A_{x,y} = \left[\,\left||x|-|y|\right|, |x|+|y|\,\right]$.

Properties 2.1. (see Rösler [13]) The signed kernel W_α is even and satisfies the following properties

$$W_\alpha(x,y,z)=W_\alpha(y,x,z)=W_\alpha(-x,z,y),$$
$$W_\alpha(x,y,z)=W_\alpha(-z,y,-x)=W_\alpha(-x,-y,-z)$$

and

$$\int_R |W_\alpha(x,y,z)|\,d\mu_\alpha(z)\le 4.$$

In the sequel we consider the signed measure $\nu_{x,y}$ on R, given by

$$\nu_{x,y}=\begin{cases} W_\alpha(x,y,z)\,d\mu_\alpha(z) & if \quad x,y\in R\setminus 0, \\ d\delta_x(z) & if \quad y=0, \\ d\delta_y(z) & if \quad x=0. \end{cases}$$

Theorem 2.1. (see Rösler [13]) (i) Let $\alpha>-1/2$ and $\lambda\in C$. The Dunkl kernel E_α satisfies the following product formula:

$$E_\alpha(\lambda x)E_\alpha(\lambda y)=\int_R E_\alpha(\lambda z)\,d\nu_{x,y}(z), \quad x,y\in R.$$

(ii) The measures $\nu_{x,y}$ have the following properties:

$$su\ pp(\nu_{x,y})=A_{x,y}\cup(-A_{x,y}), \quad \|\nu_{x,y}\|:=\int_R d\,|\nu_{x,y}|(z)\le 4.$$

Definition 2.1. For $x,y\in R$ and f a continuous function on R, we put

$$\tau_x f(y)=\int_R f(z)\,d\nu_{x,y}(z).$$

The operators $\tau_x, x\in R$, are called Dunkl translation operators on R and it can be expressed in the following form (see ref. [13])

$$\tau_x f(y)=C_\alpha\int_0^\pi f_e\left(\sqrt{x^2+y^2-2|xy|\cos\theta}\right)h_1(x,y,\theta)(\sin\theta)^{2\alpha}\,d\theta$$
$$+C_\alpha\int_0^\pi f_0\left(\sqrt{x^2+y^2-2|xy|\cos\theta}\right)h_2(x,y,\theta)(\sin\theta)^{2\alpha}\,d\theta,$$

where $f=f_e+f_0$, f_0 and f_e being respectively the odd and the even parts of f, with $C_\alpha=\Gamma(\alpha+1)/\left(\sqrt{\pi}\,\Gamma(\alpha+1/2)\right)$, $h_1(x,y,\theta)=1-\mathrm{sgn}(xy)\cos\theta$ and

$$h_2(x,y,\theta)=\begin{cases}\dfrac{(x+y)[1-\operatorname{sgn}(xy)\cos\theta]}{\sqrt{x^2+y^2-2|xy|\cos\theta}} & if \quad xy\neq 0,\\ 0 & if \quad xy=0.\end{cases}$$

Proposition 2.1. (see Mourou [8]) (i) The operator $\tau_x, x\in R$, is a continuous linear operator from $\mathrm{E}(R)$ into itself.

(ii) For all $f\in \mathrm{E}(R)$ and $x,y\in R$, we have

$$\tau_x f(y)=\tau_y f(x),\ \ \tau_0 f(x)=f(x),$$
$$\tau_x\circ\tau_y=\tau_y\circ\tau_x,\ \Lambda_\alpha\circ\tau_x=\tau_x\circ\Lambda_\alpha.$$

Proposition 2.2. (see Soltani [12]) (i) If f is an even positive continuous function, then $\tau_x f$ is positive.

(ii) For all $x\in R$ the operator τ_x extends to $L_{p,\alpha}(R), p\geq 1$ and we have for $f\in L_{p,\alpha}(R)$,

$$\|\tau_x f\|_{p,\alpha}\leq 4\|f\|_{p,\alpha}.$$

(ii) For all $x,\lambda\in R$ and $f\in L_{1,\alpha}(R)$, we have

$$\Im_\alpha(\tau_x f)(\lambda)=E_\alpha(i\lambda x)\Im_\alpha f(\lambda).$$

Let $B(x,t)=\{y\in R:|y|\in\,]\max\{0,|x|-t\},|x|+t[\}$ and $t>0$. Then $B(0,t)=\,]-t,t\,[$ and $\mu_\alpha(]-t,t\,[)=\left(2^{\alpha+1}(\alpha+1)\Gamma(\alpha+1)\right)^{-1}t^{2\alpha+2}$.

3. Some embeddings into the total D-Morrey spaces

Definition 3.1. [6] Let 1 ≤ p < ∞, 0 ≤ λ ≤ 2α + 2 and [t]1 = min{1,t}, t > 0. We denote by $L_{p,\lambda}(R)$ Morrey space (≡ D-Morrey space), bye $\widetilde{L}_{p,\lambda}(R)$ the modified Morrey space (≡ modified D-Morrey space), associated with the Dunkl operator [6, 9] and by $L_{p,\lambda,\mu}(R)$ Morrey space (≡ total D-Morrey space) as the set of locally integrable functions $f(x),\ x\in R$, with the finite norms

$$\|f\|_{L_{p\varsigma\lambda}(R)}:=\sup_{x\in R,t>0}\left(t^{-\lambda}\int_{B(0,t)}\tau_x|f|^p(y)d\mu_\alpha(y)\right)^{1/p},$$
$$\|f\|_{\widetilde{L}_{p,\lambda}(R)}:=\sup_{x\in R,t>0}\left([t]_1^{-\lambda}\int_{B(0,t)}\tau_x|f|^p(y)d\mu_\alpha(y)\right)^{1/p},$$

$$\|f\|_{L_{p,\lambda,\mu}(R)} := \sup_{x\in R, t>0}\left([t]_1^{-\lambda}[1/t]_1^{\mu}\int_{B(0,t)}\tau_x|f|^p(y)d\mu_\alpha(y)\right)^{1/p},$$

respectively.

If $\min\{\lambda,\mu\}<0$ or $\max\{\lambda,\mu\}>2\alpha+2$, then $L_{p,\lambda,\mu}(R)=\Theta(R)$, where $\Theta(R)$ is the set of all functions equivalent to 0 on R.

$$L_{p,0,0}(R)=\tilde{L}_{p,0}(R)=L_{p,0}(R)=L_p(R),$$

$$L_{p,\lambda,\lambda}(R)=L_{p,\lambda}(R),\ L_{p,\lambda,0}(R)=\tilde{L}_{p,\lambda}(R)$$

$$L_{p,\lambda,\mu}(R)\subset_{\succ} L_{p,\lambda}(R) \text{ and } \|f\|_{L_{p,\lambda}(R)}\le\|f\|_{L_{p,\lambda,\mu}(R)}, \tag{3.1}$$

$$L_{p,\lambda,\mu}(R)\subset_{\succ} L_{p,\mu}(R) \text{ and } \|f\|_{L_{p,\mu}(R)}\le\|f\|_{L_{p,\lambda,\mu}(R)}. \tag{3.2}$$

Definition 3.2. [4] Let $1 \le p < \infty$, $0 \le \lambda \le 2\alpha + 2$. We denote by $WL_{p,\lambda}(R)$ weak D-Morrey space, by $W\tilde{L}_{p,\lambda}(R)$ the modified weak D-Morrey space [6, 9] and by $WL_{p,\lambda,\mu}(R)$ weak total D-Morrey space as the set of locally integrable functions $f(x)$, $x\in R$ with finite norms

$$\|f\|_{WL_{p,\lambda}(R)} := \sup_{r>0} r \sup_{x\in R,t>0}\left(t^{-\lambda}\mu_\alpha\{y\in B(0,t):\tau_x|f|(y)>r\}\right)^{1/p},$$

$$\|f\|_{W\tilde{L}_{p,\lambda}(R)} := \sup_{r>0} r \sup_{x\in R,t>0}\left([t]_1^{-\lambda}\mu_\alpha\{y\in B(0,t):\tau_x|f|(y)>r\}\right)^{1/p},$$

$$\|f\|_{W\tilde{L}_{p,\lambda}(R)} := \sup_{r>0} r \sup_{x\in R,t>0}\left([t]_1^{-\lambda}[1/t]_1^{\mu}\mu_\alpha\{y\in B(0,t):\tau_x|f|(y)>r\}\right)^{1/p},$$

respectively.

We note that

$$L_{p,\lambda,\mu}(R)\subset WL_{p,\lambda,\mu}(R) \text{ and } \|f\|_{WL_{p,\lambda,\mu}(R)}\le\|f\|_{L_{p,\lambda,\mu}(R)}.$$

Lemma 3.1. If $0<p<\infty$, $0\le\lambda\le 2\alpha+2$ and $0\le\mu\le 2\alpha+2$, then

$$L_{p,2\alpha+2,\mu}(R)\subset_{\succ} L_\infty(R)\subset_{\succ} L_{p,\lambda,2\alpha+2}(R)$$

and

$$\|f\|_{L_{p,\lambda,2\alpha+2}(R)}\le b_k^{1/p}\|f\|_{L_\infty(R)}\le\|f\|_{L_{p,2\alpha+2,\mu}(R)}.$$

Proof. Let $f\in L_\infty(R)$. Then for all $x\in R$ and $0<t\le 1$

$$\left(t^{-\lambda}\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p}\le b_\alpha^{1/p}\|f\|_{L_\infty},\ 0\le\lambda\le 2\alpha+2$$

and for all $x\in R$ and $t\ge 1$

$$\left(t^{-2\alpha-2}\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p}\le b_\alpha^{1/p}\|f\|_{L_\infty}.$$

Therefore $f\in L_{p,\lambda,2\alpha+2}(R)$ and

$$\|f\|_{L_{p,\lambda,2\alpha+2}(R)}\le b_\alpha^{1/p}\|f\|_{L_\infty}.$$

Let $f\in L_{p,2\alpha+2,\mu}(R)$. By the Lebesgue's differentiation theorem we have (see [9, Section 2, Corolarry 2.2])

$$\lim_{t\to 0}\mu_\alpha(B(0,t))^{-1}\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)=|f(x)|^p \quad a.e.\ x\in R.$$

Then

$$|f(x)|=\left(\lim_{t\to 0}\mu_\alpha(B(0,t))^{-1}\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p}$$

$$\le b_\alpha^{1/p}\left(t^{-2\alpha-2}\int_{B(0,t)}\tau_x|f(y)|^p\,d\mu_\alpha(y)\right)^{1/p}\le b_\alpha^{1/p}\,\|f\|_{L_{p,2\alpha+2,\mu}(R)}.$$

Therefore $f\in L_\infty(R)$ and

$$\|f\|_{L_\infty(R)}\le b_\alpha^{1/p}\|f\|_{L_{p,2\alpha+2,\mu}(R)}.$$

Corollary 3.1. If $0<p<\infty$ then

$$L_{p,2\alpha+2}(R)=\tilde{L}_{p,2\alpha+2}(R)=L_\infty(R)$$

and

$$\|f\|_{L_{p,2\alpha+2}(R)}=\|f\|_{\tilde{L}_{p,2\alpha+2}(R)}=b_\alpha^{1/p}\|f\|_{L_\infty(R)}.$$

Lemma 3.2. Let $1\le p<\infty,\ 0\le\lambda\le 2\alpha+2$ and $0\le\mu\le 2\alpha+2$. Then

$$L_{p,\lambda,\mu}(R)=L_{p,\lambda}(R)\cap L_{p,\mu}(R)$$

and

$$\|f\|_{L_{p,\lambda,\mu}(R)}=\max\left\{\|f\|_{L_{p,\lambda}(R)},\|f\|_{L_{p,\mu}(R)}\right\}.$$

Proof. Let $f\in L_{p,\lambda,\mu}(R)$. Then by (3.1) and (3.2) we have

$$L_{p,\lambda,\mu}(R)\subset\!\!\!\succ L_{p,\lambda}(R)\cap L_{p,\mu}(R)$$

and

$$\max\left\{\|f\|_{L_{p,\lambda}(R)},\|f\|_{L_{p,\mu}(R)}\right\}\le\|f\|_{L_{p,\lambda,\mu}(R)}.$$

Let $f\in L_{p,\lambda}(R)\cap L_{p,\mu}(R)$. Then by Proposition 2.1 we have

$$\|f\|_{L_{p,\lambda,\mu}(R)}=\sup_{x\in R,t>0}\left([t]_1^{-\lambda}[1/t]_1^{\mu}\int_{B(0,t)}\tau_x|f|^p(y)d\mu_\alpha(y)\right)^{1/p}$$

$$=\max\left\{\sup_{x\in R,0<t\le 1}\left(t^{-\lambda}\int_{B(0,t)}\tau_x|f|^p(y)d\mu_\alpha(y)\right)^{1/p},\right.$$

$$\left.\sup_{x\in R,t>1}\left(t^{-\mu}\int_{B(0,t)}\tau_x|f|^p(y)d\mu_\alpha(y)\right)^{1/p}\right\}$$

$$\le\max\left\{\|f\|_{L_{p,\lambda}(R)},\|f\|_{L_{p,\mu}(R)}\right\}.$$

Therefore, $f\in L_{p,\lambda,\mu}(R)$ and the embedding $L_{p,\lambda}(R)\cap L_{p,\mu}(R)\subset\!\!\!\succ L_{p,\lambda,\mu}(R)$ is valid.

Thus $L_{p,\lambda,\mu}(R)=L_{p,\lambda}(R)\cap L_{p,\mu}(R)$ and

$$\|f\|_{L_{p,\lambda,\mu}(R)} = \max\{\|f\|_{L_{p,\lambda}(R)}, \|f\|_{L_{p,\mu}(R)}\}.$$

Corollary 3.1. [9, Lemma 3.2] If $0 < p < \infty,\ 0 \le \lambda \le 2\alpha + 2$, then

$$\widetilde{L}_{p,\lambda}(R) = L_{p,\lambda}(R) \cap L_p(R)$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda}(R)} = \max\{\|f\|_{L_{p,\lambda}(R)}, \|f\|_{L_p(R)}\}.$$

From Lemmas 3.1 and 3.2 for $1 \le p < \infty$ we have

$$\widetilde{L}_{p,2\alpha+2}(R) = L_\infty(R) \cap L_p(R).$$

Lemma 3.3. Let $0 < p < \infty,\ 0 \le \lambda \le 2\alpha + 2$ and $0 \le \mu \le 2\alpha + 2$. Then

$$WL_{p,\lambda,\mu}(R) = WL_{p,\lambda}(R) \cap WL_{p,\mu}(R)$$

and

$$\|f\|_{WL_{p,\lambda,\mu}(R)} = \max\{\|f\|_{WL_{p,\lambda}(R)}, \|f\|_{WL_{p,\mu}(R)}\}.$$

Corollary 3.2. If $0 < p < \infty,\ 0 \le \lambda \le 2\alpha + 2$, then

$$W\widetilde{L}_{p,\lambda}(R) = WL_{p,\lambda}(R) \cap WL_p(R)$$

and

$$\|f\|_{W\widetilde{L}_{p,\lambda}(R)} = \max\{\|f\|_{WL_{p,\lambda}(R)}, \|f\|_{WL_p(R)}\}.$$

Remark 3.1. If $0 < p < \infty$, and $\min\{\lambda, \mu\} < 0$ or $\max\{\lambda, \mu\} > 2\alpha + 2$, then

$$L_{p,\lambda,\mu}(R) = WL_{p,\lambda,\mu}(R) = \Theta(R).$$

Lemma 3.4. If $0 < p < \infty$, $0 \le \lambda_2 \le \lambda_1 \le 2\alpha + 2$ and $0 \le \mu_1 \le \mu_2 \le 2\alpha + 2$, then

$$L_{p,\lambda_1,\mu_1}(R) \subset_{\succ} L_{p,\lambda_2,\mu_2}(R)$$

and

$$\|f\|_{L_{p,\lambda_2,\mu_2}(R)} \le \|f\|_{L_{p,\lambda_1,\mu_1}(R)}.$$

Proof. Let $f \in L_{p,\lambda,\mu}(R),\ 0 < p < \infty,\ 0 \le \lambda_2 \le \lambda_1 \le 2\alpha + 2$, $0 \le \mu_1 \le \mu_2 \le 2\alpha + 2$. Then

$$\|f\|_{L_{p,\lambda_2,\mu_2}(R)} = \max\left\{ \sup_{x\in R, 0<t\le 1} \left(t^{\lambda_1-\lambda_2} t^{-\lambda_1} \int_{B(0,t)} \tau_x |f|^p(y) d\mu_\alpha(y) \right)^{1/p}, \right.$$

$$\left. \sup_{x\in R, t\ge 1} \left(t^{\mu_1-\mu_2} t^{-\mu_1} \int_{B(0,t)} \tau_x |f|^p(y) d\mu_\alpha(y) \right)^{1/p} \right\} \le \|f\|_{L^d_{p,\lambda_1,\mu_1}}.$$

On the total D-Morrey spaces the following embedding is valid.

Lemma 3.5. Let $0 \le \lambda < 2\alpha + 2$, $0 \le \mu < 2\alpha + 2$, $0 \le \beta_1 < 2\alpha + 2 - \lambda$ and $0 \le \beta_2 < 2\alpha + 2 - \mu$. Then for $\frac{2\alpha + 2 - \lambda}{\beta_1} \le p \le \frac{2\alpha + 2 - \mu}{\beta_2}$

$$L_{p,\lambda,\mu}(R) \subset_{\succ} L_{1,2\alpha+2-\beta_1,2\alpha+2-\beta_2}(R)$$

and for $f \in L_{p,\lambda,\mu}(R)$ the following inequality

$$\|f\|_{L_{1,2\alpha+2-\beta_1,2\alpha+2-\beta_2}(R)} \le b_\alpha^{\frac{1}{p'}} \|f\|_{L_{p,\lambda,\mu}(R)}$$

is valid.

Proof. Let $0<\lambda \le 2\alpha+2$, $0 \le \mu < 2\alpha+2$, $0<\beta_1<2\alpha+2-\lambda$, $0<\beta_2<2\alpha+2-\mu$, $f \in L_{p,\lambda,\mu}(R)$ and $\frac{2\alpha+2-\lambda}{\beta_1} \le p \le \frac{2\alpha+2-\mu}{\beta_2}$. By the Hölder's inequality we have

$$\|f\|_{L_{1,2\alpha+2-\beta_1,2\alpha+2-\beta_2}}(R) \le \sup_{x\in R,t>0}\left([t]_1^{\beta_1-2\alpha-2}[1/t]_1^{-\beta_2+2\alpha+2}\int_{B(0,t)}\tau_x|f|(y)d\mu_\alpha(y)\right)$$

$$\le b_\alpha^{\frac{1}{p'}} \sup_{x\in R,t>0}\left([t]_1 t^{-1}\right)^{-(2\alpha+2)/p'}[t]_1^{\beta_1-\frac{2\alpha+2-\lambda}{p}}\left([t]_1^{-\lambda}[1/t]_1^{\mu}\int_{B(0,t)}\tau_x|f|^p(y)d\mu_\alpha(y)\right)^{1/p}$$

$$\le b_\alpha^{\frac{1}{p'}}\|f\|_{L_{p,\lambda,\mu}(R)}\sup_{t>0}\left([t]_1 t^{-1}\right)^{\frac{2\alpha+2-\mu}{p}-\beta_2}[t]_1^{\beta_1-\frac{2\alpha+2-\lambda}{p}}.$$

Note that

$$\sup_{t>0}\left([t]_1 t^{-1}\right)^{\frac{2\alpha+2-\mu}{p}-\beta_2}[t]_1^{\beta_1-\frac{2\alpha+2-\lambda}{p}} = \max\left\{\sup_{0<t\le 1} t^{\beta_1-\frac{2\alpha+2-\lambda}{p}},\ \sup_{t>1} t^{\beta_2-\frac{2\alpha+2-\mu}{p}}\right\}<\infty$$

if and only if $\frac{2\alpha+2-\lambda}{\beta_1} \le p \le \frac{2\alpha+2-\mu}{\beta_2}$.

Therefore $f \in L_{1,2\alpha+2-\beta_1,2\alpha+2-\beta_2}(R)$ and

$$\|f\|_{L_{1,2\alpha+2-\beta_1,2\alpha+2-\beta_2}(R)} \le b_\alpha^{\frac{1}{p'}} \|f\|_{L_{p,\lambda,\mu}(R)}.$$

Corollary 3.3. [9,Lemma 3.4] Let $0 \le \lambda < 2\alpha+2$ and $0 \le \beta < 2\alpha+2-\lambda$. Then for $p = \frac{2\alpha+2-\lambda}{\beta}$

$$L_{p,\lambda}(R) \subset_{\succ} L_{1,2\alpha+2-\beta}(R)$$

and for $f \in L_{p,\lambda}(R)$ the following inequality

$$\|f\|_{L_{1,2\alpha+2-\beta}(R)} \le b_\alpha^{\frac{1}{p'}} \|f\|_{L_{p,\lambda}(R)}$$

is valid.

Corollary 3.4. [9,Lemma 3.5] Let $0 \le \lambda < 2\alpha+2$ and $0 \le \beta < 2\alpha+2-\lambda$. Then for $\frac{2\alpha+2-\lambda}{\beta} \le p \le \frac{2\alpha+2-\mu}{\beta}$

$$\widetilde{L}_{p,\lambda}(R) \subset_{\succ} \widetilde{L}_{1,2\alpha+2-\beta}(R)$$

and for $f \in \widetilde{L}_{p,\lambda}(R)$ the following inequality

$$\|f\|_{\widetilde{L}_{1,2\alpha+2-\beta}(R)} \leq b_{\alpha}^{\frac{1}{p'}} \|f\|_{L_{p,\lambda}(R)}$$

is valid.

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References

1. Dunkl C.F. Integral kernels with reflection group invariance, Can. J. Math., V.43, 1991, pp.1213-1227.
2. Dunkl C.F. Hankel transforms associated to finite reflection groups, Contemp. Math., V.138, 1992, pp.123-138.
3. Guliyev V.S. Maximal commutator and commutator of maximal function on total Morrey spaces, J. Math. Inequal., V.16, N.4, 2022, pp.1509-1524.
4. Guliyev V.S., Mammadov Y.Y. On fractional maximal function and fractional integral associated with the Dunkl operator on the real line, J. Math. Anal. Appl., V.353, N.1, 2009, pp.449-459.
5. de Jeu M.F.E. The Dunkl transform, Invent. Math., V.113, 1993, pp.147-162.
6. Mammadov Y.Y. Some embeddings into the Morrey spaces associated with the Dunkl operator on R. Proc. of NASA, Embedding theorems, Harmonic analysis, N.13, 2007, pp.258-269.
7. Morrey C.B. On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc., V.43, 1938, pp.126-166.
8. Mourou M.A. Transmutation operators associated with a Dunkl-type differential-difference operator on the real line and certain of their applications, Integral Transforms Spec. Funct., V.12, N.1, 2001, pp.77-88.
9. Muslumova F.A. On embeddings into the Morrey and modified Morrey spaces in the Dunkl setting, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics, V.40, N.1, 2020, pp.135-145.
10. Omarova M.N. Commutator of anisotropic maximal function with BMO functions on anisotropic total Morrey spaces, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics, V.43, N.1, 2023, pp.1-15.
11. Sifi M., Soltani F. Generalized Fock spaces and Weyl relations for the Dunkl kernel on the real line, J. Math. Anal. Appl., V.270, 2002, pp.92-106.
12. Soltani F. L_p-Fourier multipliers for the Dunkl operator on the real line, J. Funct. Anal., V.209, 2004, pp.16-35.

13. Rösler M. Bessel-type signed hypergroups on R, in: H. Heyer, A. Mukherjea (Eds.), Probability Measures on Groups and Related Structures XI, Proceedings of Oberwolfach, 1994, pp.292-304.
14. Trimeche K. Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators, Integral Transforms Spec. Funct., V.13, 2002, pp.17-38.
15. Watson G.N. A Treatise on Theory of Bessel Functions, Cambridge University Press, Cambridge, 1966.